Quantum Field Theory

Set 5: solutions

Exercise 1:

We want to compute the Noether charge associated to rotations for a massive vector field. Lorentz transformations act as usual:

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \simeq x^\mu + w^\mu_{\ \nu} x^\nu = x^\mu - \frac{1}{2}\epsilon^\mu_{\alpha\beta} w^{\alpha\beta} \implies \epsilon^\mu_{\alpha\beta} = -\left(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha\right),$$
$$A'_\rho(x') = \Lambda_\rho^{\ \nu} A_\nu(x) = \Lambda_\rho^{\ \nu} A_\nu(\Lambda^{-1}x') \simeq \left(\delta^\nu_\rho + w_\rho^{\ \nu}\right) A_\nu(x'^\mu - w^\mu_{\ \sigma} x'^\sigma) \simeq A_\rho(x') + w_\rho^{\ \nu} A_\nu(x') - w^{\mu\nu} x'_\nu \partial'_\mu A_\rho(x').$$

Dropping the primes on x' we have:

$$A'_\rho(x) - A_\rho(x) = \frac{1}{2} w^{\alpha\beta} \Delta_{\rho,\alpha\beta} \implies \Delta_{\rho,\alpha\beta} = \left(\eta_{\rho\alpha} A_\beta - \eta_{\rho\beta} A_\alpha\right) + \left(x_\alpha \partial_\beta - x_\beta \partial_\alpha\right) A_\rho.$$

Notice that we have defined $\epsilon^\mu_{\alpha\beta}$ and $\Delta_{\rho,\alpha\beta}$ without the factor $1/2$. It is clear that all the definitions are equivalent, as long as they are all consistent, however this is the choice that provides the correct normalization of the generators of rotations: $[J_i, J_j] = i\epsilon_{ijk} J_k$. The Noether current is then:

$$J^\mu_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \Delta_{\rho,\alpha\beta} - \epsilon^\mu_{\alpha\beta} \mathcal{L} = -F^{\mu\rho} \Delta_{\rho,\alpha\beta} + \left(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha\right) \mathcal{L}$$
$$= -F^{\mu\rho}\left(\eta_{\rho\alpha} A_\beta + x_\alpha \partial_\beta A_\rho\right) + F^{\mu\rho}\left(\eta_{\rho\beta} A_\alpha + x_\beta \partial_\alpha A_\rho\right) + \left(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha\right) \mathcal{L}.$$

We can now focus on the current associated to rotations: $(\alpha\beta) = (ij)$. In addition we take $\mu = 0$ in order to obtain the charge:

$$Q_{ij} \equiv \int d^3x\, J^0_{ij} = -\left(\int d^3x \left(F^0_{\ i} A_j + F^{0m} x_i \partial_j A_m\right) - (i \leftrightarrow j)\right) = \int d^3x \left(-F_{0i} A_j + F_{0m} x_i \partial_j A_m\right) - (i \leftrightarrow j),$$

where we have lowered all the indices. We finally define the three generators of rotations as:

$$J_k = \frac{1}{2}\epsilon_{ijk} Q_{ij} = \epsilon_{ijk} \int d^3x \left(-F_{0i} A_j + F_{0m} x_i \partial_j A_m\right).$$

From now on we will keep all indices down. Recalling the definition of the conjugate momenta ($\Pi^i = -F^{0i}$) we can also rewrite:

$$J_k = \epsilon_{ijk} \int d^3x \left(\Pi_i A_j - \Pi_m\, x_i \partial_j A_m\right).$$

One can check that this definition is correctly normalized.

By intuition we can guess the form of the angular momentum. This is most easily done by splitting it in the orbital part L_i and the spin part S_i:

$$J_i = L_i + S_i.$$

Since we are dealing with a free theory, Noether currents are quadratic are quadratic in terms of ladder operators, i.e. they have the form $\sim aa^\dagger$; indeed operators of the kind $\sim (a)^{m+n}(a^\dagger)^m$ and $(a)^m(a^\dagger)^{m+n}$, $n > m$, would change the number of particles, while we expect the number of particles to commute with the angular momentum, while $\sim (a)^m(a^\dagger)^m$ operators give non vanishing contributions only on $n \geq m$ particle states, while in a free theory we expect the total angular momentum to be just the sum of the angular momentum of all particles, i.e. a sum of single particle contributions. Finally we must also require that both J_i is a a pseudovector, i.e. that it transforms as a vector under rotations and does not change sign under parity.

The orbital part of the angular momentum should depend only on the motion and not on the internal structure of the fields. Then we expect it to have the same form of the free scalar field:

$$L_k = i\epsilon_{ijk} B \int d\Omega_{\vec{k}}\, a_m(\vec{k}) \left(k_i \frac{\partial}{\partial k^j} \right) a_m^\dagger(\vec{k}), \tag{1}$$

where B is an unknown coefficient. For the spin part we also require that no term proportional to k_i or $\frac{\partial}{\partial k^j}$ appears, since we expect its value to be independent of the momentum of the states on which it acts. Then the only vector operator we can write which satisfies all the requirements is:

$$S_k = i\epsilon_{ijk} A \int d\Omega_{\vec{k}}\, a_i(\vec{k}) a_j^\dagger(\vec{k}), \tag{2}$$

where A is an unknown coefficient.
As a first simple check one can verify that $PJ_iP = J_i$, where P is the parity operator. We recall that:

$$Pa_i(\vec{k})P = -\eta_p a_i(\vec{k}_P), \qquad Pa_i^\dagger(\vec{k})P = -\eta_P a_i^\dagger(\vec{k}_P), \qquad \eta_P^2 = 1.$$

Then one can check, using $P^2 = 1$,

$$PS_kP = i\epsilon_{ijk} A \int d\Omega_{\vec{k}}\, Pa_i(\vec{k})PPa_j^\dagger(\vec{k})P = i\epsilon_{ijk} A \int d\Omega_{\vec{k}}\, a_i(\vec{k}_P)a_j^\dagger(\vec{k}_P) = i\epsilon_{ijk}\eta_P^2 A \int d\Omega_{\vec{k}_P}\, a_i(\vec{k}_P)a_j^\dagger(\vec{k}_P) = S_k,$$

$$PL_kP = i\epsilon_{ijk} B \int d\Omega_{\vec{k}}\, Pa_m(\vec{k})P \left(k_i \frac{\partial}{\partial k^j} \right) Pa_m^\dagger(\vec{k})P = i\epsilon_{ijk}\eta_P^2 Bs \int d\Omega_{\vec{k}}\, a_m(\vec{k}_P) \left(k_i \frac{\partial}{\partial k^j} \right) a_m^\dagger(\vec{k}_p) = L_k,$$

where in both expressions we have changed the integration variable to $\vec{k}_P$.
To fix the coefficients, we have to impose the commutation rules:

$$[J_i, J_j] = i\epsilon_{ijk} J_k. \tag{3}$$

To make our task easier, we will impose the stronger condition:

$$[S_i, S_j] = i\epsilon_{ijk} S_k \qquad \textbf{and} \qquad [L_i, L_j] = i\epsilon_{ijk} L_k. \tag{4}$$

This is physically reasonable given the different nature of orbital and spin angular momentum. Formally this can be justified noticing that for our guess $[S_i, L_j] = 0$. Let's prove this:

$$[S_k, L_l] = -\epsilon_{ijk}\epsilon_{npl} AB \int d\Omega_{\vec{k}} d\Omega_{\vec{p}} [a_i(\vec{k})a_j^\dagger(\vec{k}), a_m(\vec{p}) p_n \frac{\partial}{\partial p_p} a_m^\dagger(\vec{p})].$$

Using $[AB, CD] = A[B, C]D + CA[B, D] + [A, C]BD + C[A, D]B$ and the commutation rules of creation and annihilation operators

$$[a_i(\vec{k}), a_j^\dagger(\vec{p})] = (2\pi)^3 2\omega_{\vec{k}} \delta_{ij} \delta^3(\vec{p} - \vec{k}) \equiv f(\vec{k})\delta_{ij}\delta^3(\vec{p} - \vec{k}),$$

where we have defined $f(\vec{k}) \equiv (2\pi)^3 2\omega_{\vec{k}}$ for convenience, we can evaluate the commutator:

$$\begin{aligned}
&[a_i(\vec{k})a_j^\dagger(\vec{k}), a_m(\vec{p}) p_n \frac{\partial}{\partial p_p} a_m^\dagger(\vec{p})] \\
&= -\delta_{mj} f(\vec{k})\delta^3(\vec{k} - \vec{p}) a_i(\vec{k}) \left(p_n \frac{\partial}{\partial p_p} \right) a_m^\dagger(\vec{p}) + \delta_{mi} f(\vec{k}) a_m(\vec{p}) a_j^\dagger(\vec{k}) \left(p_n \frac{\partial}{\partial p_p} \delta^3(\vec{k} - \vec{p}) \right) \\
&= -f(\vec{k})\delta^3(\vec{k} - \vec{p}) a_i(\vec{k}) \left(p_n \frac{\partial}{\partial p_p} \right) a_j^\dagger(\vec{p}) + f(\vec{k}) a_i(\vec{p}) a_j^\dagger(\vec{k}) \left(p_n \frac{\partial}{\partial p_p} \delta^3(\vec{k} - \vec{p}) \right).
\end{aligned}$$

These two terms under integration cancel. Indeed, recalling $d\Omega_{\vec{k}} = \frac{d^3k}{f(\vec{k})}$, the first gives:

$$-\int d\Omega_{\vec{k}} d\Omega_{\vec{p}} f(\vec{k})\delta^3(\vec{k} - \vec{p}) a_i(\vec{k}) \left(p_n \frac{\partial}{\partial p_p} \right) a_j^\dagger(\vec{p}) = -\int d\Omega_{\vec{p}}\, a_i(\vec{p}) \left(p_n \frac{\partial}{\partial p_p} \right) a_j^\dagger(\vec{p}),$$

while, integrating twice by parts, the second is

$$\int d\Omega_{\vec{k}} d\Omega_{\vec{p}} f(\vec{k}) a_i(\vec{k}) a_j^\dagger(\vec{p}) \left(p_n \frac{\partial}{\partial p_p} \delta^3(\vec{k}-\vec{p}) \right) = \int \frac{d^3k\, d^3p}{f(\vec{p})} a_i(\vec{p}) a_j^\dagger(\vec{k}) \left(p_n \frac{\partial}{\partial p_p} \delta^3(\vec{k}-\vec{p}) \right)$$
$$= -\int d^3k\, d^3p\, \delta^3(\vec{k}-\vec{p}) \frac{\partial}{\partial p_p} \left(a_i(\vec{p}) \frac{p_n}{f(\vec{p})} \right) a_j^\dagger(\vec{k}) = -\int d^3p \frac{\partial}{\partial p_p} \left(a_i(\vec{p}) \frac{p_n}{f(\vec{p})} \right) a_j^\dagger(\vec{p}) \tag{5}$$
$$= \int d\Omega_{\vec{p}}\, a_i(\vec{p}) \left(p_n \frac{\partial}{\partial p_p} \right) a_j^\dagger(\vec{p}).$$

The two terms cancel and thus $[S_i, J_j] = 0$. Then the only way to realize the commutation rules (3) is to impose (4).
Now we fix A from the commutation rules of S_i:

$$[S_i, S_j] = -\epsilon_{kli}\epsilon_{mnj} A^2 \int d\Omega_{\vec{k}} d\Omega_{\vec{p}} [a_k(\vec{k}), a_l^\dagger(\vec{k}), a_m(\vec{p}) a_n^\dagger(\vec{p})]$$
$$= -\epsilon_{kli}\epsilon_{mnj} A^2 \int d\Omega_{\vec{k}} d\Omega_{\vec{p}} \left[-a_k(\vec{k}) a_n^\dagger(\vec{p}) \delta_{lm} f(\vec{k}) \delta^3(\vec{k}-\vec{p}) + a_m(\vec{p}) a_l^\dagger(\vec{k}) \delta_{kn} f(\vec{k}) \delta^3(\vec{k}-\vec{p}) \right]$$
$$= -\epsilon_{kli}\epsilon_{mnj} A^2 \int d\Omega_{\vec{k}} \left[-a_k(\vec{k}) a_n^\dagger(\vec{k}) \delta_{lm} + a_m(\vec{k}) a_l^\dagger(\vec{k}) \delta_{kn} \right].$$

Using the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ twice, we find:

$$[S_i, S_j] = A^2 \int d\Omega_{\vec{k}} \left[a_j(\vec{k}) a_i^\dagger(\vec{k}) - a_i(\vec{k}) a_j^\dagger(\vec{k}) \right].$$

We could proceed in full generality, but it is slightly easier to choose $i = 1$, $j = 2$; then comparing the r.h.s with eq. (2), we see:

$$[S_1, S_2] = A^2 \int d\Omega_{\vec{k}} \left[a_2(\vec{k}) a_1^\dagger(\vec{k}) - a_1(\vec{k}) a_2^\dagger(\vec{k}) \right] = iA\, S_3.$$

Requiring the correct commutation rule, we fix A:

$$[S_1, S_2] = iS_3 \quad \Longrightarrow \quad A = 1. \tag{6}$$

Finally we repeat the same steps for L_k. We have:

$$[L_i, L_j] = -\epsilon_{mni}\epsilon_{stj} B^2 \int d\Omega_{\vec{k}} d\Omega_{\vec{p}} [a_k(\vec{k}) k_m \frac{\partial}{\partial k^n} a_k^\dagger(\vec{k}), a_l(\vec{p}) p_s \frac{\partial}{\partial p_t} a_l^\dagger(\vec{p})].$$

The commutator is evaluated as those before:

$$[a_k(\vec{k}) k_m \frac{\partial}{\partial k^n} a_k^\dagger(\vec{k}), a_l(\vec{p}) p_s \frac{\partial}{\partial p_t} a_l^\dagger(\vec{p})]$$
$$= -a_k(\vec{k}) k_m \frac{\partial}{\partial k^n} \left(f(\vec{p}) \delta^3(\vec{k}-\vec{p}) \delta_{kl} \right) p_s \frac{\partial}{\partial p_t} a_l^\dagger(\vec{p}) + a_l(\vec{p}) p_s \frac{\partial}{\partial p_t} \left(f(\vec{k}) \delta^3(\vec{k}-\vec{p}) \delta_{kl} \right) k_m \frac{\partial}{\partial k^n} a_k^\dagger(\vec{k})$$
$$= -a_k(\vec{k}) k_m \frac{\partial}{\partial k^n} \left(f(\vec{p}) \delta^3(\vec{k}-\vec{p}) \right) p_s \frac{\partial}{\partial p_t} a_k^\dagger(\vec{p}) + a_k(\vec{p}) p_s \frac{\partial}{\partial p_t} \left(f(\vec{k}) \delta^3(\vec{k}-\vec{p}) \right) k_m \frac{\partial}{\partial k^n} a_k^\dagger(\vec{k})$$

Repeating the same steps we did in (5), we obtain:

$$\int d\Omega_{\vec{k}} d\Omega_{\vec{p}}\, a_k(\vec{k}) k_m \frac{\partial}{\partial k^n} \left(f(\vec{p}) \delta^3(\vec{k}-\vec{p}) \right) p_s \frac{\partial}{\partial p_t} a_k^\dagger(\vec{p}) = \int d\Omega_{\vec{k}}\, a_k(\vec{k}) k_m \frac{\partial}{\partial k^n} k_s \frac{\partial}{\partial k^t} a_k^\dagger(\vec{k}),$$

$$\int d\Omega_{\vec{k}} d\Omega_{\vec{p}}\, a_k(\vec{p}) p_s \frac{\partial}{\partial p_t} \left(f(\vec{k}) \delta^3(\vec{k}-\vec{p}) \right) k_m \frac{\partial}{\partial k^n} a_k^\dagger(\vec{k}) = \int d\Omega_{\vec{p}}\, a_k(\vec{p}) p_s \frac{\partial}{\partial p_t} p_m \frac{\partial}{\partial p_n} a_k^\dagger(\vec{p}).$$

Relabelling the integration variable in one of the two expressions, the commutator reads:

$$[L_i, L_j] = \int d\Omega_{\vec{k}} \left[-a_k(\vec{k}) k_m \frac{\partial}{\partial k^n} k_s \frac{\partial}{\partial k^t} a_k^\dagger(\vec{k}) + a_k(\vec{k}) k_s \frac{\partial}{\partial k^t} k_m \frac{\partial}{\partial k^n} a_k^\dagger(\vec{k}) \right].$$

In both terms inside the parenthesis, one derivative can act either on a momentum, either on the operator $a^\dagger_k(\vec{k})$. However in the second case the contribution of the two terms cancel and we are left with[1]

$$
\begin{aligned}
[L_i, L_j] &= -\epsilon_{mni}\epsilon_{stj}B^2 \int d\Omega_{\vec{k}} \left[\delta_{ns} a_k(\vec{k}) k_m \frac{\partial}{\partial k^t} a^\dagger_k(\vec{k}) - \delta_{mt} a_k(\vec{k}) k_s \frac{\partial}{\partial k^n} a^\dagger_k(\vec{k})\right] \\
&= B^2 \int d\Omega_{\vec{k}} \left[a_k(\vec{k}) k_i \frac{\partial}{\partial k^j} a^\dagger_k(\vec{k}) - a_k(\vec{k}) k_j \frac{\partial}{\partial k^i} a^\dagger_k(\vec{k})\right],
\end{aligned}
$$

where we have used $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$. Again, we specialize to $i = 1$, $j = 2$. Then comparison with (1) shows:

$$
[L_1, L_2] = B^2 \int d\Omega_{\vec{k}} \left[a_k(\vec{k}) k_1 \frac{\partial}{\partial k^2} a^\dagger_k(\vec{k}) - a_k(\vec{k}) k_2 \frac{\partial}{\partial k^1} a^\dagger_k(\vec{k})\right] = -iBL_3.
$$

Finally we can fix B via the commutation rules (4):

$$
[L_1, L_2] = iL_3 \qquad \Longrightarrow \qquad B = -1. \tag{7}
$$

Exercise 1: Explicit computation

We now proceed and expand the above expressions in terms of raising and lowering operators. Recall the definitions:

$$
\begin{aligned}
\Pi_i(x) &= \int \frac{d^3k}{(2\pi)^3}\, e^{ikx}\Pi_i(\vec{k}, t) \qquad A_i(x) = \int \frac{d^3k}{(2\pi)^3}\, e^{ikx} A_i(\vec{k}, t), \\
\Pi_i(\vec{k}) &= \frac{i}{2}\left[\left(a_\perp(\vec{k}) - a^\dagger_\perp(-\vec{k})\right)_i + \frac{M}{\omega_k}\left(a_L(\vec{k}) - a^\dagger_L(-\vec{k})\right)_i\right], \\
A_j(\vec{k}) &= \frac{1}{2\omega_k}\left[\left(a_\perp(\vec{k}) + a^\dagger_\perp(-\vec{k})\right)_j + \frac{\omega_k}{M}\left(a_L(\vec{k}) + a^\dagger_L(-\vec{k})\right)_j\right],
\end{aligned}
$$

where $\omega_k = \sqrt{k^2 + M^2}$ and we have decomposed the operator in transverse and longitudinal pieces:

$$
\begin{aligned}
a_{\perp\, i}(\vec{k}) &= P^\perp_{ij} a_j(\vec{k}), \qquad & a_{L\, i}(\vec{k}) &= P^L_{ij} a_j(\vec{k}), \\
P^\perp_{ij} &= \delta_{ij} - \frac{k_i k_j}{k^2}, \qquad & P^L &= 1 - P^\perp.
\end{aligned}
$$

In the appendix we compute the expression of the angular momentum in terms of the operators $a(k)$ explicitly. The result has the simple form:

$$
J_k = i\epsilon_{ijk} \int d\Omega_{\vec{k}} \left\{a_i(\vec{k}) a^\dagger_j(\vec{k}) - a_m(\vec{k})\left(k_i \frac{\partial}{\partial k^j}\right) a^\dagger_m(\vec{k})\right\}.
$$

The above expression is composed of two pieces, corresponding to the intrinsic spin carried by a state and the orbital angular momentum. One can look for eigenstates of the 3rd component J_3. In the reference frame in which the particle is at rest this must correspond to the spin of the state. Let us consider the states:

$$
|3\rangle = a^\dagger_3(0)|0\rangle, \qquad |\pm\rangle = \left(\frac{a^\dagger_1(0) \pm i a^\dagger_2(0)}{\sqrt{2}}\right)|0\rangle.
$$

Applying J_3 yields:

$$
\begin{aligned}
J_3|\pm\rangle &= i\int d\Omega_{\vec{k}} \left\{a_1(k) a^\dagger_2(k) - a_2(k) a^\dagger_1(k)\right\}\left(\frac{a^\dagger_1(0) \pm i a^\dagger_2(0)}{\sqrt{2}}\right)|0\rangle = \pm\left(\frac{a^\dagger_1(0) \pm i a^\dagger_2(0)}{\sqrt{2}}\right)|0\rangle = \pm|\pm\rangle, \\
J_3|0\rangle &= 0.
\end{aligned}
$$

[1] Note $\frac{\partial}{\partial k^i} k_j = \eta_{ij} = -\delta_{ij}$.

Angular momentum in terms of creation operators

Let us start from the first piece:

$$
\begin{aligned}
&\epsilon_{ijk}\int d^3x\,\Pi_i A_j \\
&= \epsilon_{ijk}\int d^3x\int \frac{d^3k_1}{(2\pi)^3}\,\frac{d^3k_2}{(2\pi)^3}\,\Pi_i(\vec{k}_1)A_j(\vec{k}_2)e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \\
&= \epsilon_{ijk}\int \frac{d^3k_1}{(2\pi)^3}\,\frac{d^3k_2}{(2\pi)^3}\,\Pi_i(\vec{k}_1)A_j(\vec{k}_2)(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2) = \epsilon_{ijk}\int\frac{d^3k}{(2\pi)^3}\,\Pi_i(-\vec{k})A_j(\vec{k}) \\
&= \epsilon_{ijk}\int \frac{d^3k}{(2\pi)^3}\,\frac{i}{4w}\left[\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_i\left(a_\perp(\vec{k})+a_\perp^\dagger(-\vec{k})\right)_j+\frac{M}{w}\left(a_L(-\vec{k})-a_L^\dagger(\vec{k})\right)_i\left(a_\perp(\vec{k})+a_\perp^\dagger(-\vec{k})\right)_j\right. \\
&\qquad\left.+\frac{w}{M}\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_i\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_j+\left(a_L(-\vec{k})-a_L^\dagger(\vec{k})\right)_i\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_j\right].
\end{aligned}
$$

Keeping on expanding we get:

$$
\begin{aligned}
&\epsilon_{ijk}\int d^3x\,\Pi_i A_j \\
&= \epsilon_{ijk}\int\frac{d^3k}{(2\pi)^3}\,\frac{i}{4w}\left[\left(a_{\perp i}(-\vec{k})a^\dagger_{\perp j}(-\vec{k})-a^\dagger_{\perp i}(\vec{k})a_{\perp j}(\vec{k})\right)+\frac{M}{w}\left(a_{Li}(-\vec{k})a^\dagger_{\perp j}(-\vec{k})-a^\dagger_{Li}(\vec{k})a_{\perp j}(\vec{k})\right)\right. \\
&\qquad\left.+\frac{w}{M}\left(a_{\perp i}(-\vec{k})a^\dagger_{Lj}(-\vec{k})-a^\dagger_{\perp i}(\vec{k})a_{Lj}(\vec{k})\right)+\left(a_{Li}(-\vec{k})a^\dagger_{Lj}(-\vec{k})-a^\dagger_{Li}(\vec{k})a_{Lj}(\vec{k})\right)\right]
\end{aligned}
$$

The additional terms we haven't written are in some case odd in k, such that $\epsilon_{ijk}a_{\perp i}(\vec{k})a_{\perp j}(-\vec{k})$ or they will cancel out with other terms coming from other pieces. We don't need to keep track of them because all these terms consist of two a or two $a^\dagger$ and we know a priori that they must cancel. The final expression for the Noether charge must contain only terms linear in a and $a^\dagger$ (it cannot mix states with different particle content) because, since it is a conserved quantity, it commutes with the Hamiltonian, so it is diagonal in the basis of multiparticle states $|\vec{k}_1,...,\vec{k}_n\rangle$ where H is diagonal.
We can finally rewrite the above expression as:

$$
\epsilon_{ijk}\int\frac{d^3k}{(2\pi)^3}\,\frac{i}{4w}\left[2a_{\perp i}(\vec{k})a^\dagger_{\perp j}(\vec{k})+2a_{Li}(\vec{k})a^\dagger_{Lj}(\vec{k})+\left(\frac{M}{w}+\frac{w}{M}\right)\left(a_{Li}(\vec{k})a^\dagger_{\perp j}(\vec{k})+a_{\perp i}(\vec{k})a^\dagger_{Lj}(\vec{k})\right)\right]. \tag{8}
$$

The second piece is:

$$
\begin{aligned}
&-\epsilon_{ijk}\int d^3x\;\Pi_m\,x_i\partial_j A_m \\
&= -\epsilon_{ijk}\int d^3x\int\frac{d^3k_1}{(2\pi)^3}\,\frac{d^3k_2}{(2\pi)^3}\Pi_m(\vec{k}_1)A_m(\vec{k}_2)e^{i\vec{k}_1\cdot\vec{x}}\left(x_i\partial_j e^{i\vec{k}_2\cdot\vec{x}}\right) \\
&= -\epsilon_{ijk}\int d^3x\int\frac{d^3k_1}{(2\pi)^3}\,\frac{d^3k_2}{(2\pi)^3}\Pi_m(\vec{k}_1)A_m(\vec{k}_2)e^{i\vec{k}_1\cdot\vec{x}}\left(k_{2j}\frac{\partial}{\partial k_2^i}e^{i\vec{k}_2\cdot\vec{x}}\right)
\end{aligned}
$$

where we have used the usual trick:

$$
\frac{\partial}{\partial x^j}e^{ik_mx_m} = -ik_je^{ik_mx_m}, \qquad\qquad i\frac{\partial}{\partial k^i}e^{ik_mx_m} = x_ie^{ik_mx_m}.
$$

Thus, integrating over d^3x we get:

$$
-\epsilon_{ijk}\int d^3x\;\Pi_m\,x_i\partial_jA_m = -\int\frac{d^3k_1}{(2\pi)^3}\,d^3k_2\,\Pi_m(\vec{k}_1)A_m(\vec{k}_2)\epsilon_{ijk}k_{2j}\frac{\partial}{\partial k_2^i}\delta^3(\vec{k}_1+\vec{k}_2).
$$

We can integrate by parts and get:

$$
\begin{aligned}
-\epsilon_{ijk}\int d^3x\;\Pi_m\,x_i\partial_jA_m &= \int\frac{d^3k_1}{(2\pi)^3}\,d^3k_2\,\Pi_m(\vec{k}_1)\delta^3(\vec{k}_1+\vec{k}_2)\left(\epsilon_{ijk}k_{2j}\frac{\partial}{\partial k_2^i}\right)A_m(\vec{k}_2) \\
= -\int\frac{d^3k}{(2\pi)^3}\,\Pi_m(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)A_m(-\vec{k}).
\end{aligned}
$$

Notice that the minus in the last equality arises by changing the order of the indices i, j. This time Π and A are contracted in a scalar product (but there is a differential operator in the middle). Rewriting the expressions in terms of ladder operators will give four contributions which we consider one by one. The first one is:

$$\int \frac{d^3k}{(2\pi)^3}\frac{-i}{4w}\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_m\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)\left(a_\perp(\vec{k})+a_\perp^\dagger(-\vec{k})\right)_m=\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{2w}a_{\perp m}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a_{\perp m}^\dagger(\vec{k}).$$

Recalling that $a_{\perp m}(\vec{k})=P^\perp_{mr}a_r(\vec{k})$ we have:

$$=\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{2w}a_{\perp m}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a_m^\dagger(\vec{k})+\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{2w}a_{\perp m}(\vec{k})a_r(\vec{k})^\dagger\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)P^\perp_{mr}.$$

We need the following expression:

$$\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)P^\perp_{mr}=\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)\left(\delta_{mr}-\frac{k_mk_r}{k^2}\right)=\epsilon_{imk}P^L_{ir}+\epsilon_{irk}P^L_{im},$$

where we must take care of the relation $\frac{\partial k_i}{\partial k^j}=-\delta_{ij}$. Hence, substituting this gives:

$$\int \frac{d^3k}{(2\pi)^3}\frac{-i}{2w}a_{\perp m}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a_m^\dagger(\vec{k})+\int \frac{d^3k}{(2\pi)^3}\frac{-i}{2w}\epsilon_{imk}a_{\perp m}(\vec{k})a_{Li}^\dagger(\vec{k}). \tag{9}$$

The second contribution is:

$$\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{4w}\frac{M}{w}\left(a_L(-\vec{k})-a_L^\dagger(\vec{k})\right)_m\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)\left(a_\perp(\vec{k})+a_\perp^\dagger(-\vec{k})\right)_m.$$

This time the piece where the derivative acts on a_m is absent since we would get a $P^\perp$ acting on a_L which gives zero. Hence we only have:

$$\begin{aligned}
&=\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{4w}\frac{M}{w}\left(a_L(-\vec{k})-a_L^\dagger(\vec{k})\right)_m\left(a(\vec{k})+a^\dagger(-\vec{k})\right)_r\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)P^\perp_{mr}\\
&=\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{4w}\frac{M}{w}\left\{\epsilon_{imk}\left(a_L(-\vec{k})-a_L^\dagger(\vec{k})\right)_m\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_i+\epsilon_{irk}\left(a_L(-\vec{k})-a_L^\dagger(\vec{k})\right)_i\left(a(\vec{k})+a^\dagger(-\vec{k})\right)_r\right\}\\
&=\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{4w}\frac{M}{w}\left\{\epsilon_{imk}\left(a_{Li}(\vec{k})a_{\perp m}^\dagger(\vec{k})+a_{\perp i}(\vec{k})a_{Lm}^\dagger(\vec{k})\right)\right\}.
\end{aligned} \tag{10}$$

The third contribution is:

$$\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{4w}\frac{w}{M}\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_m\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_m.$$

Also here we find the piece where the derivative acts on a_m to be absent since we would get a P^L acting on $a_\perp$ which is also zero. Hence the only contribution is

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{4w}\frac{w}{M}\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_m\left(a(\vec{k})+a^\dagger(-\vec{k})\right)_r\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)P^L_{mr}\\
&=\int \frac{d^3k}{(2\pi)^3}\,\frac{i}{4w}\frac{w}{M}\left\{\epsilon_{imk}\left(a_\perp(-\vec{k})-a_\perp^\dagger(\vec{k})\right)_m\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_i\right\}\\
&=\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{4w}\frac{w}{M}\left\{\epsilon_{imk}\left(a_{\perp i}(\vec{k})a_{Lm}^\dagger(\vec{k})+a_{Li}(\vec{k})a_{\perp m}^\dagger(\vec{k})\right)\right\}.
\end{aligned} \tag{11}$$

Finally, the last term:

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{4w}\left(a_L(-\vec{k})-a_L^\dagger(\vec{k})\right)_m\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)\left(a_L(\vec{k})+a_L^\dagger(-\vec{k})\right)_m=\int \frac{d^3k}{(2\pi)^3}\frac{-i}{2w}a_{Lm}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a_{Lm}^\dagger(\vec{k})\\
&=\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{2w}a_{Lm}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a_m^\dagger(\vec{k})+\int \frac{d^3k}{(2\pi)^3}\,\frac{i}{2w}\left\{\epsilon_{imk}a_{Lm}(\vec{k})a_{Li}^\dagger(\vec{k})+\epsilon_{imk}a_{Li}(\vec{k})a_m^\dagger(\vec{k})\right\}\\
&=\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{2w}a_{Lm}(\vec{k})\left(\epsilon_{ijk}k_i\frac{\partial}{\partial k^j}\right)a_m^\dagger(\vec{k})+\int \frac{d^3k}{(2\pi)^3}\,\frac{-i}{2w}\left\{\epsilon_{imk}a_{Lm}(\vec{k})a_{\perp i}^\dagger(\vec{k})\right\}.
\end{aligned} \tag{12}$$

Collecting formulas (8),(9),(10),(11),(12), we find:

$$J_k = \epsilon_{ijk} \int \frac{d^3k}{(2\pi)^3} \frac{i}{2w} \Big\{ a_{\perp i}(\vec{k}) a^\dagger_{\perp j}(\vec{k}) + a_{Li}(\vec{k}) a^\dagger_{Lj}(\vec{k}) + a_{Li}(\vec{k}) a^\dagger_{\perp j}(\vec{k}) + a_{\perp i}(\vec{k}) a^\dagger_{Lj}(\vec{k})$$
$$- (a_{Lm}(\vec{k}) + a_{\perp m}(\vec{k})) \left(k_i \frac{\partial}{\partial k^j} \right) a^\dagger_m(\vec{k}) \Big\}$$
$$= i\epsilon_{ijk} \int d\Omega_{\vec{k}} \left\{ a_i(\vec{k}) a^\dagger_j(\vec{k}) - a_m(\vec{k}) \left(k_i \frac{\partial}{\partial k^j} \right) a^\dagger_m(\vec{k}) \right\}.$$

Exercise 2

The angular momentum for a spin-1 massive free field is given by (see exercise 1):

$$J_k = i\epsilon_{ijk} \int d\Omega_{\vec{k}} \left\{ a_i(\vec{k}) a^\dagger_j(\vec{k}) - a_m(\vec{k}) \left(k_i \frac{\partial}{\partial k^j} \right) a^\dagger_m(\vec{k}) \right\}.$$

The second part vanishes when acting on particles at rest, then we can isolate the spin:

$$S_k = i\epsilon_{ijk} \int d\Omega_{\vec{k}} a_i(\vec{k}) a^\dagger_j(\vec{k}).$$

That this is the generator of the internal (coordinate independent) part of rotations is confirmed by its action on creation operators $a^\dagger_i(\vec{p})$:

$$[S_i, a^\dagger_j(\vec{p})] = i\epsilon_{mni} \left[\int d^3\Omega_{\vec{k}} a_m(\vec{k}) a^\dagger_n(\vec{k}), a^\dagger_j(\vec{p}) \right] = i\epsilon_{mni} \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[a_m(\vec{k}), a^\dagger_j(\vec{p}) \right] a^\dagger_n(\vec{k})$$
$$= i\epsilon_{mni} \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} (2\pi)^3 2\omega_{\vec{k}} \delta^3(\vec{k} - \vec{p}) \delta_{mj}\, a^\dagger_n(\vec{k}) = i\epsilon_{jni} a^\dagger_n(\vec{p}) = i\epsilon_{ijn} a^\dagger_n(\vec{k})$$

This shows that $\vec{S}$ rotates the creation operators $a^\dagger_i(\vec{k})$ between them, as it expected for the spin.
The operators $a^\dagger_i(\vec{k})$ create massive spin-1 particles with momentum $\vec{k}$, then the most general massive spin-1 single particle state in the rest frame is written as:

$$|\vec{\alpha}\rangle \equiv \alpha_i a^\dagger_i(\vec{0}) |0\rangle = \vec{\alpha} \cdot \vec{a}^\dagger(\vec{0}) |0\rangle .$$

Given the previous result and the fact that the vacuum is rotation invariant, $J_k |0\rangle = S_k |0\rangle = 0$, we see that the action of S_k on such a state is:

$$S_k |\vec{\alpha}\rangle = \alpha_i \left[S_k, a^\dagger_i(\vec{0}) \right] |0\rangle = i\epsilon_{kij} \alpha_i a^\dagger_j(\vec{0}) |0\rangle .$$

In particular for the z component we find:

$$S_3 \left(\alpha_i a^\dagger_i(\vec{0}) |0\rangle \right) = i\epsilon_{3ji} \alpha_j a^\dagger_i(\vec{0}) |0\rangle = i\alpha_1 a^\dagger_2 |0\rangle - i\alpha_2 a^\dagger_1 |0\rangle .$$

We can now diagonalize S_3 on such states. Since a basis for the states $|\vec{\alpha}\rangle$ is given simply by $\{a^\dagger_i(\vec{0}) |0\rangle\}$, the equation

$$S_3 |\vec{\alpha}^a\rangle = \lambda_a |\vec{\alpha}^a\rangle \qquad \text{(no sum over } a\text{)},$$

is equivalent to the matrix eigenvalue problem:

$$(i\epsilon_{3ji}\alpha_j - \alpha_i) a^\dagger_i(\vec{0}) |0\rangle = 0 \qquad \Longleftrightarrow \qquad \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \lambda \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

The eigenvalues of such a problem correspond to $\lambda = 1, 0, -1$, as it is to be expected for the spin of a spin 1 particle. The eigenvectors of this system are easily found:

$$\lambda = 1 \quad \longrightarrow \quad \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{pmatrix}, \qquad \lambda = 0 \quad \longrightarrow \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda = -1 \quad \longrightarrow \quad \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}.$$

Finally, let us compute $S^2 = S_i S_i = 2$ on single particle states:

$$S^2 |\vec{\alpha}\rangle = S_k S_k |\vec{\alpha}\rangle = S_k i\epsilon_{kij}\alpha_i a_j^\dagger(\vec{0})|0\rangle = -\epsilon_{kjl}\epsilon_{kij}\alpha_i a_l^\dagger |0\rangle = 2\delta_{li}\alpha_i a_l^\dagger |0\rangle = 2|\vec{\alpha}\rangle .$$

Here we used $\epsilon_{kjl}\epsilon_{kij} = -2\delta_{li}$. We expect $S^2 = \ell(\ell+1)$ on a spin ℓ state, hence this result confirms that $|\vec{\alpha}\rangle$ is a spin 1 state.

Exercise 3

The momentum $p^\mu = (E, 0, 0, p)$ and the polarization vector $\varepsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0)$ satisfy the Lorentz-invariant constraint $p^\mu \varepsilon_\mu = 0$, in addition to the normalization conditions $\varepsilon^\mu \varepsilon^*_\mu = -1$ and $p^\mu p_\mu = M^2$.
ε^μ is an eigenvector of helicity with eigenvalue $+1$, as can be seen recalling the helicity operator:

$$h \equiv \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|} = J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and by applying it on $\vec{\epsilon} \equiv (1, i, 0)$.
After a transverse boost in the y direction:

$$\Lambda = \begin{pmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we find:

$$\begin{aligned} p'^\mu &= (\gamma E, 0, \gamma\beta E, p)\,, \\ \varepsilon'^\mu &= \frac{1}{\sqrt{2}}(i\gamma\beta, 1, i\gamma, 0)\,. \end{aligned}$$

Note that, correctly, $p'^\mu \varepsilon'_\mu = 0$.
In order to decompose this vector on a basis of vectors with definite helicity, it is convenient to first rotate the three space in such a way as to align the new z direction to $\vec{p}'$, namely to perform the transformation:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{p}{\gamma k} & -\frac{\beta E}{k} \\ 0 & 0 & \frac{\beta E}{k} & \frac{p}{\gamma k} \end{pmatrix},$$

where $k \equiv \gamma^{-1}\sqrt{p^2 + (\gamma\beta E)^2}$. So we get:

$$\begin{aligned} \tilde{p}^\mu &= \gamma\,(E, 0, 0, k)\,, \\ \tilde{\varepsilon}^\mu &= \frac{1}{\sqrt{2}}\left(i\gamma\beta, 1, \frac{ip}{k}, \frac{i\gamma\beta E}{k}\right). \end{aligned}$$

The helicity basis is a set of polarization vectors $\tilde{\varepsilon}_{(i)}$ with definite helicity; they satisfy the transversality condition $\tilde{\varepsilon}^\mu_{(i)}\tilde{p}_\mu = 0, \ \ \forall\, i = -, 0, +$. In this frame they are:

$$\begin{aligned} \tilde{\varepsilon}^\mu_{(+)} &= \frac{1}{\sqrt{2}}(0, 1, i, 0)\,, \\ \tilde{\varepsilon}^\mu_{(-)} &= \frac{1}{\sqrt{2}}(0, 1, -i, 0)\,, \\ \tilde{\varepsilon}^\mu_{(0)} &= \frac{\gamma}{M}(k, 0, 0, E)\,, \end{aligned}$$

where the subscripts indicate the helicity eigenvalues.
Decomposing $\tilde{\varepsilon}'^\mu$ on this basis yields:

$$\tilde{\varepsilon}^\mu = \left(\frac{1 + p/k}{2}\right)\tilde{\varepsilon}^\mu_{(+)} + \left(\frac{1 - p/k}{2}\right)\tilde{\varepsilon}^\mu_{(-)} + \left(\frac{i\beta M}{\sqrt{2}k}\right)\tilde{\varepsilon}^\mu_{(0)}.$$

Note in particular that starting from a massive vector with positive helicity and performing a transverse boost, results in a superposition of all possible helicity states. This is different from the case of a massless vector. Indeed, it has been proven in Set17 (and it can be deduced here as well by taking the limit $M \to 0$) that for the massless case, starting with a positive helicity state, we end up with a positive helicity state (plus a longitudinal component).

Exercise 4

In general, a state with n-particles and m-antiparticles can be expressed as the superposition of eigenstates of the momentum:

$$|\Phi\rangle = \int d\Omega_{\vec{p}_1}...d\Omega_{\vec{p}_n}\, d\Omega_{\vec{q}_1}...d\Omega_{\vec{q}_m} f(\vec{p}_1,...,\vec{p}_n,\vec{q}_1,...,\vec{q}_m) a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)\, b^\dagger(\vec{q}_1)...b^\dagger(\vec{q}_m)|0\rangle.$$

In the simple case of a system consisting of a particle and an anti-particle in the center of mass ($\vec{p}_1 = -\vec{q}_1$) with a defined angular momentum l we have:

$$|\Phi_l\rangle = \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p}) a^\dagger(\vec{p}) b^\dagger(-\vec{p})|0\rangle,$$

where $f_l(\vec{p},-\vec{p})$ is the wave function describing a state with a given angular momentum (it is actually a superposition of spherical harmonics with total angular momentum l) and satisfies the property:

$$f_l(\vec{p},-\vec{p}) = (-1)^l f_l(-\vec{p},\vec{p})\ .$$

Let us now perform a parity transformation: in general each particle acquires a multiplicative phase η_P but since the antiparticle gets the same factor η_P and $\eta_P^2 = 1$ this factor never appears. In addition to this, the spatial momenta are inverted:

$$\begin{aligned} P|\Phi_l\rangle &= \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p})\, P a^\dagger(\vec{p}) P^\dagger\, P b^\dagger(-\vec{p}) P^\dagger|0\rangle \\ &= \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p})\, a^\dagger(-\vec{p})\, b^\dagger(\vec{p})|0\rangle \\ &= \int d\Omega_{\vec{p}}\ f_l(-\vec{p},\vec{p})\, a^\dagger(\vec{p})\, b^\dagger(-\vec{p})|0\rangle = (-1)^l|\Phi_l\rangle, \end{aligned}$$

where in the first line we have inserted $P^\dagger P = 1$ and we have used the invariance of the vacuum $P\,|0\rangle = |0\rangle$. Note also that $P^\dagger = P$, since we require that acting twice with parity has to be equal to the identity transformation, thus $POP^\dagger = P^\dagger OP$ for any operator O. Therefore a state made of a scalar particle-antiparticle pair with a given angular momentum changes by a factor $(-1)^l$ under parity.
Let's now consider a state consisting of a fermionic particle-antiparticle pair. We can write such a state as:

$$|\Psi_{l,S}\rangle = \sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p}) \chi_S(r,t) \tilde{d}^\dagger(\vec{p},r) b^\dagger(-\vec{p},t)|0\rangle,$$

where the two functions satisfy:

$$f_l(\vec{p},-\vec{p}) = (-1)^l f_l(-\vec{p},\vec{p})\,, \qquad \chi_S(t,r) = (-1)^{S+1}\chi_S(r,t).$$

Notice that the transformation property for the spin function $\chi_S(r,t)$ reflects the fact that the product of two spin 1/2 states is symmetric if the total spin is 1 and is antisymmetric if the total spin is 0. Again we can apply the parity operator:

$$\begin{aligned} P|\Psi_{l,S}\rangle &= \sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p})\, \chi_S(r,t)\, P\tilde{d}^\dagger(\vec{p},r) P^\dagger\, P b^\dagger(-\vec{p},t) P^\dagger|0\rangle \\ &= -\sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p})\, \chi_S(r,t)\, \tilde{d}^\dagger(-\vec{p},r)\, b^\dagger(\vec{p},t)\,|0\rangle = (-1)^{l+1}|\Psi_{l,S}\rangle. \end{aligned}$$

Notice that P doesn't touch the spins.